THE · 35TH · PEE – DEE · REGIONAL · HIGH – SCHOOL
M A T H E M A T I C S · T O U R N A M E N T

Written Competition

S P O N S O R E D · B Y · F R A N C I S · M A R I O N · U N I V E R S I T Y
A N D · T H E · P E E · D E E · E D U C A T I O N · C E N T E R
TUESDAY · 2011 · DECEMBER · 06

Instructions

Do not turn over this page until instructed to do so.

Neatly print (not sign) your name *as you wish it to appear if you are given an award.*

During the competition, no calculators are allowed. Cellphones also are strictly prohibited.

Each final answer must be placed in its proper answer box or it will not be scored.

Because the judges must score over 250 papers in under an hour, they have not time to deal with unsimplified answers. Therefore:

	Unacceptable	Acceptable
One must perform all arithmetic that evaluates to an integer.	$2^2 \cdot 3^3 \cdot 5$	540
One must cancel all common factors in fractions of two integers.	4/6	2/3
In writing fractions, one must choose *either* an integer over an integer *or* a mixed fraction with largest possible whole part.	$2 + \dfrac{5}{3}$	$\dfrac{11}{3}$ or $3 + \dfrac{2}{3}$
In writing square-roots, one must "take out" all perfect squares.	$\sqrt{24}$	$2\sqrt{6}$
One must rationalize the denominator whenever a square-root appears in the bottom of a fraction. After rationalization, one must also be sure to cancel any common factors.	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{2}}{2}$ or $\dfrac{1}{2}\sqrt{2}$
	$\dfrac{2}{\sqrt{7}-1}$	$\dfrac{\sqrt{7}+1}{3}$
All ratios must be written as a pure number in conventional notation. Translate ":" as "/" and, if necessary, simplify the resulting fraction.	$1:2$	$\dfrac{1}{2}$

—— *For official use only* ——

↑ Name. (Print neatly and fully.)

↑ High School.   * Used only in tie-breaking.

Awards

|	
Page **1**. (# 1, 2, 3)	Page **2**. (# 4, 5, 6)
Page **3**. (# 7, 8, 9)	Page **4**. (# 10, 11, 12)
Page **5**. (# 13, 14, 15)	Page **6**. (# 16, 17, 18)
Total Correct	Weighted Sum*

1. A cube whose sides are 1 unit long has a sphere inscribed inside it. What is the radius of the sphere?

2. What is the volume of the sphere in Problem 1? *Be sure to simplify any fractions and do not approximate π but let it stand in the answer.*

3. Simplify, working base 10: $\dfrac{111{,}111}{11}$
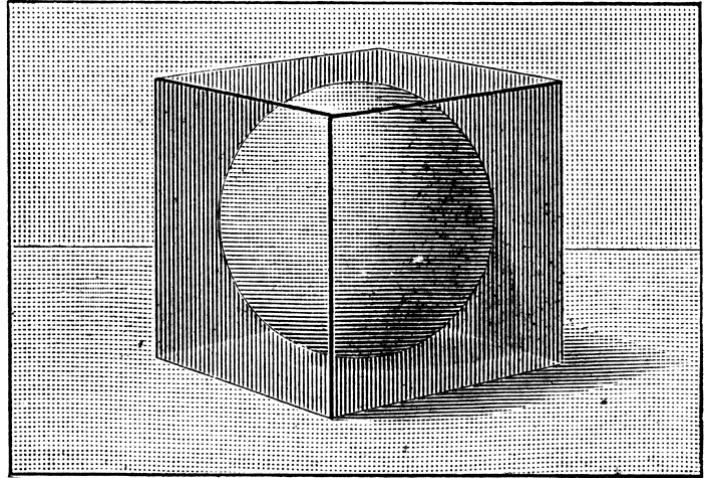
— In order to receive credit, answers must appear in these boxes and be properly simplified. —

Answer to Problem 1:	Answer to Problem 2:	Answer to Problem 3:
units	cubic units	base ten

4. Simplify, working base 2: $\dfrac{111{,}111}{11}$

5. Two circles, each of radius 3 units, have centers that are 12 units apart. What is the length of a line segment internally tangent to the two circles?

6. Mary is writing down all the numbers from 1 to 1000. She is very careful in her penmanship, and takes 1 second to write each numeral. The number 5 takes her 1 second to write, the number 37 takes her 2 seonds to write, and the number 1000 takes her four seconds. How long will it take Mary to write all the numbers from 1 to 1000 this neatly? *To gain credit for this problem, you must also answer in minutes and seconds.*
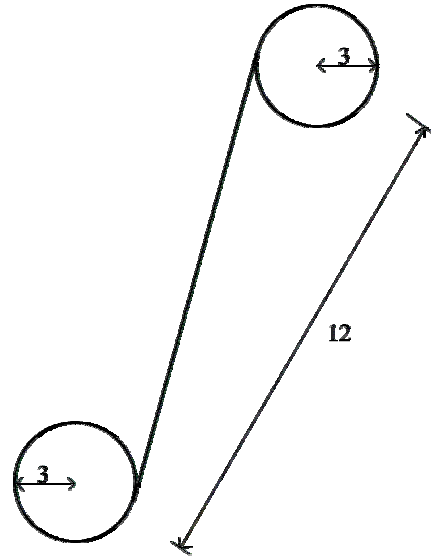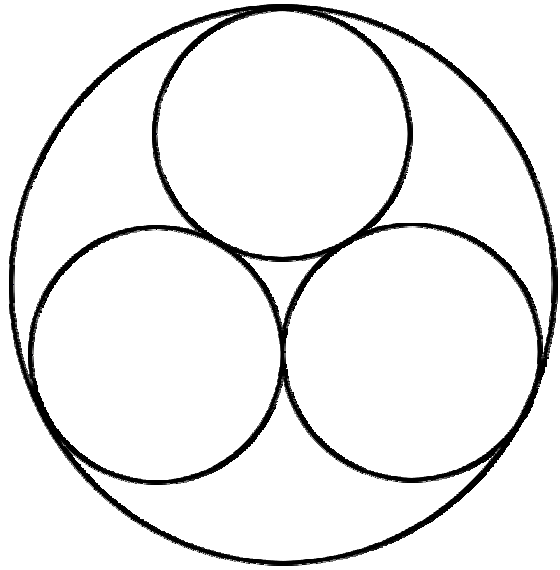
— In order to receive credit, answers must appear in these boxes and be properly simplified. —

Answer to Problem 4:	Answer to Problem 5:	Answer to Problem 6:
base two	units	minutes + seconds

7. Simplify completely the following quantity, which represents the sum of the golden ratio and its reciprocal:

$$\frac{\sqrt{5}-1}{2} + \frac{2}{\sqrt{5}-1}$$

8. Three circles, each of radius 1 unit, are tangent to one another. A larger circle is externally tangent to the first three. What is the radius of the larger circle?

9. Simplify thoroughly using various trigonometric identities:

$$\frac{\sin^2 x + \cos^2 x - \tan^2 x - \cot^2 x + \sec^2 x + \csc^2 x}{\sin^2 x \cdot \cos^2 x \cdot \sec^2 x \cdot \csc^2 x \cdot \tan^2 x \cdot \cot^2 x}$$

— In order to receive credit, answers must appear in these boxes and be properly simplified. —

Answer to Problem 7:	Answer to Problem 8:	Answer to Problem 9:
	units	

The following very brief table of logarithms may be of assistance in answering some of the questions on this page. All results are good to five significant digits of accuracy.

|		
$\log_{10}(1) = 0.00000$	$\log_{10}(4) = 0.60206$	$\log_{10}(7) = 0.84510$
$\log_{10}(2) = 0.30103$	$\log_{10}(5) = 0.69897$	$\log_{10}(8) = 0.90309$
$\log_{10}(3) = 0.47712$	$\log_{10}(6) = 0.77815$	$\log_{10}(9) = 0.95424$

10. How many digits are in the integer 2^{2016} when written out base 10? *Note well: the problem is exactly worded and must be answered exactly. Fractional answers cannot be right.*

11. What is the *first* digit of the number 2^{2016}?

12. What is the *last* digit of the number 2^{2016}?

— *In order to receive credit, answers must appear in these boxes and be properly simplified.* —

Answer to Problem 10:	Answer to Problem 11:	Answer to Problem 12:

13. A regular octagon has an eight-pointed star within it, drawn by every vertex being joined to the vertex three vertices away, as shown. What is the angle at each point of the star, such as $\angle DAF$ in the picture? *You may answer either in degrees or radians.*
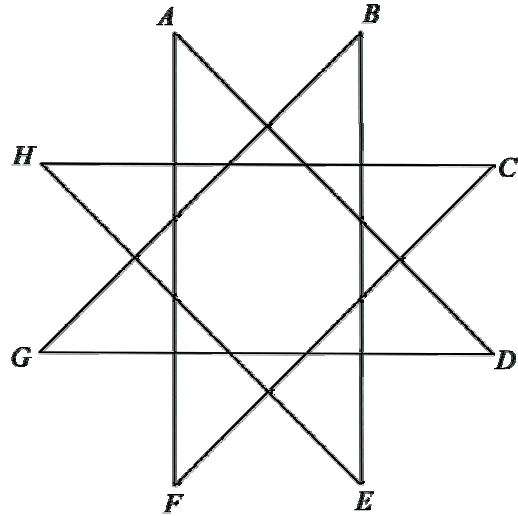
14. A regular octagon has an eight-pointed star drawn within it as before. The outer octagon has sides of length 1 unit. (That is, the line segment *AB*, if drawn, would have length 1 unit.) What is the length of one of the sides of the star, such as the length of the line segment *AD* in the picture?

15. What is the area of the part of the plane covered by the eight-pointed star described in the previous problem?

— In order to receive credit, answers must appear in these boxes and be properly simplified. —

Answer to Problem 13:	Answer to Problem 14:	Answer to Problem 15:
	units	square units

16. Determine the following square-root, where both question and answer are in base 2:

$$\sqrt{10101001_{\text{base two}}} = ?_{\text{base two}}$$

17. What is the volume of the cube in Problem 18?

18. A sphere of radius 1 unit has a cube inscribed within it. What is the length of the side of the cube? (Sphere only is shown. Inscribed cube is inside it.)
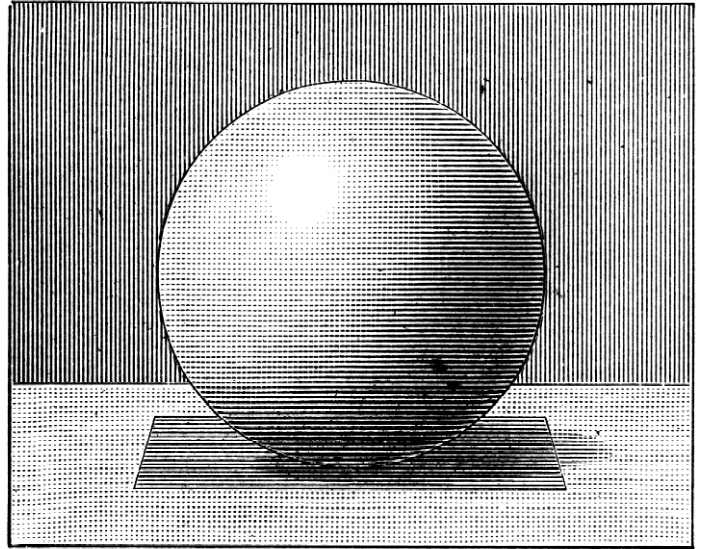
Note well: *Remember throughout to perform all simplifications indicated on the front of this competition and to have answers in the proper boxes. The judges are instructed to reject answers that do not adhere to these protocols even if they are otherwise right.*

Answer to Problem 16:	Answer to Problem 17:	Answer to Problem 18:
base two	cubic units	units

Answer to Ultimate Tiebreaker: